Everything you always wanted to know about the Levy-stable law, but were afraid to ask

Rafał Weron
Topics

- Introduction
  - The STABLE books
  - A bit of history
- Properties of stable laws
- Computer simulation of stable variables
- Estimation of parameters
- Other interesting topics
The STABLE books

Gnedenko–Kolmogorov (1954)

SIMULATION AND CHAOTIC BEHAVIOR OF α-STABLE STOCHASTIC PROCESSES

Aleksander Janicki
Aleksander Weron

(1994)

Stable Distributions
Models for Heavy Tailed Data
John P. Nolan
(2003)

MODERN PROBABILITY AND STATISTICS

Chance and Stability
Stable Distributions and Their Applications
Vladimir V. Uchaikin and Vladlim Zolotarev

(1999)

STABLE NON-GAUSSIAN RANDOM PROCESSES
Stochastic Models with Infinite Variance
Gennady Samorodnitsky
Murad S. Taqqu

(1994)

(c) 2002 by Rafał Weron
A bit of history

- Cauchy (~1850) extended the theory of errors, generalizing the Gaussian formula to
  \[ f_N(x) = \frac{1}{\pi} \int_0^\infty \exp(-ct^N) \cos(tx) \, dt \]
  he succeeded in evaluating the integral only for \( N=1 \)

- Bernstein (1919) observed that \( f_N \) is positive definite (and hence a pdf) only when \( 0 < N \leq 2 \)

- **Paul Levy (1924)** studied sums of independent variables; found a generalization of the CLT
  - i.i.d. + finite variance \( \Rightarrow \) Gaussian
  - **Levy**: i.i.d. \( \Rightarrow \) stable
  - Levy-Khinchin: i.d. \( \Rightarrow \) infinitely-divisible (eg. NIG, hyperbolic laws)
A bit of applications history

- Holtsmark (1915) – gravitational field of stars (3/2-stable)

- **Finance**

- **Signal processing**

- **Statistical physics** (*⇒ Paretian/power-law tails*)
Topics

- Introduction
- Properties of stable laws
  - General properties
  - Stability under summation
  - Characteristic function representations
  - Power-law tails
- Computer simulation of stable variables
- Estimation of parameters
- Other interesting topics
General properties

- A four parameter family: $S_\alpha(\sigma, \beta, \mu)$
- PDF’s in “closed form” for
  - $\alpha=2$ (Gaussian, normal)
  - $\alpha=1$ (Cauchy)
  - $\alpha=0.5$ (Levy, $|\beta|=1$)
- For $\alpha<2$ variance is infinite ($EX^p<\infty$ only for $p<\alpha$)
- $\sigma$ - scale parameter
- $\mu$ - location parameter; $\mu=EX$ for $\alpha>1$
General properties: $\alpha$

- Index of stability $\alpha \in (0,2]$ (tail index, tail/characteristic exponent)

PDF’s for $S_\alpha(1,0,0)$
General properties: $\beta$

- Skewness parameter $\beta \in [-1,1]$

PDF’s for $S_{0.8}(1,\beta,0)$
General properties: $\alpha$ and $\beta$

PDF’s for $S_\alpha(1,0.5,0)$
Stability under summation

Proposition 2.1 \( If \ X_i \sim S_\alpha(\sigma_i, \beta_i, \mu_i) \) for \( i = 1, 2 \) are independent random variables, then

\[
X_1 + X_2 \sim S_\alpha(\sigma, \beta, \mu),
\]

with

\[
\sigma = (\sigma_1^\alpha + \sigma_2^\alpha)^{1/\alpha}, \quad \beta = \frac{\beta_1\sigma_1^\alpha + \beta_2\sigma_2^\alpha}{\sigma_1^\alpha + \sigma_2^\alpha}, \quad \mu = \mu_1 + \mu_2.
\]

- Summation scheme holds true only for \( X_i \)'s with the same \( \alpha \)'s
- Otherwise, the sum is infinitely divisible but not stable
Characteristic function representations

- **Standard representation** $S_\alpha(\sigma, \beta, \mu)$

\[
\log \phi(t) = \begin{cases} 
-\sigma^\alpha |t|^\alpha \{1 - i\beta \text{sign}(t) \tan \frac{\pi \alpha}{2}\} + i\mu t, & \alpha \neq 1, \\
-\sigma |t| \{1 + i\beta \text{sign}(t) \frac{2}{\pi} \log |t|\} + i\mu t, & \alpha = 1.
\end{cases}
\]

- **Zolotarev’s M representation** $S^0_\alpha(\sigma, \beta, \mu_0)$

\[
\log \phi_0(t) = \begin{cases} 
-\sigma^\alpha |t|^\alpha \{1 + i\beta \text{sign}(t) \tan \frac{\pi \alpha}{2} [(|t|^{1-\alpha} - 1)]\} + i\mu_0 t, & \alpha \neq 1, \\
-\sigma |t| \{1 + i\beta \text{sign}(t) \frac{2}{\pi} \log (|t|)\} + i\mu_0 t, & \alpha = 1.
\end{cases}
\]

- PDF is continuous in all 4 parameters, $\sigma$ and $\mu_0$ are scale and location parameters, i.e. $\sigma X + \mu_0 \sim S^0_\alpha(\sigma, \beta, \mu_0)$ for $X \sim S_\alpha(1, \beta, 0)$
Characteristic function representations cont.

$S_\alpha(\sigma,\beta,\mu)$

$S^0_\alpha(\sigma,\beta,\mu)$
Power-law tails: Theory

If $X \sim S_{\alpha<2}(1, \beta, 0)$ then

$$P(X > x) = 1 - F(x) \rightarrow C_\alpha (1 + \beta) x^{-\alpha},$$

$$P(X < -x) = F(-x) \rightarrow C_\alpha (1 - \beta) x^{-\alpha},$$

$$C_\alpha = \left(2 \int_0^\infty x^{-\alpha} \sin x \, dx \right)^{-1} = \frac{1}{\pi} \Gamma(\alpha) \sin \frac{\pi \alpha}{2}.$$
Power-law tails: Rate of convergence

Convergence to power-law tail varies with $\alpha$
Power-law tails: Empirical analysis for $\alpha=1.8$

10^4 samples

10^6 samples

(c) 2002 by Rafał Weron
Power-law tails:
Empirical analysis for $\alpha=1.95$
Topics

- Introduction
- Properties of stable laws
- Computer simulation of stable variables
  - Zolotarev’s integral representations
  - The Chambers-Mallows-Stuck method
- Estimation of parameters
- Other interesting topics
Zolotarev’s integral representations

- Zolotarev’s B representation $S^2_\alpha(\sigma_2, \beta_2, \mu)$

\[
\log \phi(t) = \begin{cases} 
-\sigma^\alpha_2 |t|^\alpha \exp\{-i\beta_2 \text{sign}(t) \frac{\pi}{2} K(\alpha)\} + i\mu t, & \alpha \neq 1, \\
-\sigma_2 |t| \left\{ \frac{\pi}{2} + i\beta_2 \text{sign}(t) \log |t| \right\} + i\mu t, & \alpha = 1,
\end{cases}
\]

- with $K(\alpha) = \alpha - 1 + \text{sign}(1 - \alpha)$
- Relation between $S_\alpha(\sigma, \beta, \mu)$ and $S^2_\alpha(\sigma_2, \beta_2, \mu)$ for $\alpha \neq 1$:

\[
\tan \left(\beta_2 \frac{\pi K(\alpha)}{2}\right) = \beta \tan \frac{\pi \alpha}{2}, \quad \sigma_2 = \sigma \left(1 + \beta^2 \tan^2 \frac{\pi \alpha}{2}\right)^{1/(2\alpha)}
\]

- For $\alpha = 1$: $\beta_2 = \beta$ and $\sigma_2 = \frac{2}{\pi} \sigma$
Zolotarev’s integral representations cont.

(Zolotarev 1986, Remark 1, p. 78). CDF $F(x, \alpha, \beta_2)$ of $S^2_\alpha$
if $\alpha \neq 1$ and $x > 0$ then

$$F(x, \alpha, \beta_2) = C(\alpha, \beta_2) + \frac{\epsilon(\alpha)}{\pi} \int_{\gamma_0}^{\pi/2} \exp[-x^{\alpha/(\alpha-1)} U_\alpha(\gamma, \gamma_0)] d\gamma,$$

if $\alpha = 1$ and $\beta_2 > 0$ then

$$F(x, 1, \beta_2) = \frac{1}{\pi} \int_{-\pi/2}^{\pi/2} \exp[-e^{-x/\beta_2} U_1(\gamma, \beta_2)] d\gamma.$$
Zolotarev's integral representations cont.

where

\[ \epsilon(\alpha) = \text{sign}(1 - \alpha), \quad \gamma_0 = -\frac{\pi}{2} \beta_2 \frac{K(\alpha)}{\alpha}, \]

\[ C(\alpha, \beta_2) = 1 - \frac{1}{4} (1 + \beta_2 K(\alpha)/\alpha)(1 + \epsilon(\alpha)), \]

\[ U_\alpha(\gamma, \gamma_0) = \left( \frac{\sin \alpha(\gamma - \gamma_0)}{\cos \gamma} \right)^{\alpha/(1-\alpha)} \cos(\gamma - \alpha(\gamma - \gamma_0)) \cos \gamma, \]

\[ U_1(\gamma, \beta_2) = \frac{\pi}{2} + \beta_2 \gamma \exp \left( \frac{1}{\beta_2} \left( \frac{\pi}{2} + \beta_2 \gamma \right) \tan \gamma \right). \]
Chambers-Mallows-Stuck method

- Zolotarev (1956-1966), Chernin-Ibragimov (1959)
- Kanter (1975) – method for simulating $S_{\alpha<1}(1,1,0)$
- Chambers-Mallows-Stuck (1976)
  - Symmetric $\alpha$-stable (S\(\alpha\)S) case (for $\alpha=2$ reduces to the Box-Muller method for Gaussian r.v.):
    - generate a random variable $V$ uniformly distributed on $(-\frac{\pi}{2}, \frac{\pi}{2})$ and an independent exponential random variable $W$ with mean 1;
    - compute
      \[
      X = \frac{\sin(\alpha V)}{(\cos(V))^{1/\alpha}} \times \left(\frac{\cos(V - \alpha V)}{W}\right)^{(1-\alpha)/\alpha}.
      \] (3.8)
Chambers–Mallows–Stuck method

explicit proof in Weron (1996)

- generate a random variable $V$ uniformly distributed on $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$ and an independent exponential random variable $W$ with mean 1;

- for $\alpha \neq 1$ compute:

$$X = S_{\alpha, \beta} \times \frac{\sin(\alpha(V + B_{\alpha, \beta}))}{(\cos(V))^{1/\alpha}} \times \left(\frac{\cos(V - \alpha(V + B_{\alpha, \beta}))}{W}\right)^{(1-\alpha)/\alpha}, \quad (3)$$

where

$$B_{\alpha, \beta} = \frac{\arctan(\beta \tan \frac{\pi \alpha}{2})}{\alpha},$$

$$S_{\alpha, \beta} = \left[1 + \beta^2 \tan^2 \frac{\pi \alpha}{2}\right]^{1/(2\alpha)};$$

- for $\alpha = 1$ compute:

$$X = \frac{2}{\pi} \left[\left(\frac{\pi}{2} + \beta V\right) \tan V - \beta \log \left(\frac{\pi W \cos V}{\frac{\pi}{2} + \beta V}\right)\right]. \quad (4)$$
Chambers–Mallows–Stuck method: comments

- If \( X \sim S_{\alpha}(1, \beta, 0) \) then \( Y \sim S_{\alpha}(\sigma, \beta, \mu) \), where

\[
Y = \begin{cases} 
\sigma X + \mu, & \alpha \neq 1, \\
\sigma X + \frac{2}{\pi} \beta \sigma \log \sigma + \mu, & \alpha = 1,
\end{cases}
\]

- Other approaches
  - Mantegna (1994) – using Bergstrom (1952) series expansion
Topics

- Introduction
- Properties of stable laws
- Computer simulation of stable variables
- **Estimation of parameters**
  - Estimation of the tail index ($\alpha$)
  - Quantile methods
  - Characteristic function based methods
  - Maximum Likelihood Method
- Other interesting topics
Estimation of the tail index $\alpha$

- Hill (1975)
  $$\alpha_{Hill}(k) = \left( \frac{1}{k} \sum_{n=1}^{k} \log \frac{X(n)}{X(k+1)} \right)^{-1}$$
  where $X(n)$ is the $n$-th order statistics

- Other methods
  - Pickand (1975)
  - Dekkers-Einmahl-DeHaan (1989)

- Optimal choice of $k$ (usually in the vicinity of the plateau)
  - Beirlant-Vynckier-Teugels (1996)
Performance of the Hill estimator – $10^4$ samples

\[ \alpha_{\text{Hill}} \]

\[ \alpha = 1.95 \]

\[ \alpha = 1.8 \]

\[ \alpha = 1.5 \]

\[ \alpha = 1 \]

Order statistics

(c) 2002 by Rafał Weron
Performance of the Hill estimator – $10^6$ samples

\begin{align*}
\alpha_{\text{Hill}} & \quad \alpha = 1.95 \\
\alpha_{\text{Hill}} & \quad \alpha = 1.8
\end{align*}

Order statistics

(c) 2002 by Rafał Weron
Sample quantile methods

Let $x_f$ be the $f$-th population quantile, so that $S_\alpha(\sigma, \beta, \mu)(x_f) = f$. Let $\hat{x}_f$ be the corresponding sample quantile, i.e. $\hat{x}_f$ satisfies $F_n(\hat{x}_f) = f$.

- Fama-Roll (1968, 1971) $\hat{\sigma} = \frac{\hat{x}_{0.72} - \hat{x}_{0.28}}{1.654}$

$$S_\alpha\left(\frac{\hat{x}_f - \hat{x}_{1-f}}{2\hat{\sigma}}\right) = f = 0.95, 0.96, 0.97$$

- McCulloch (1986) – estimators of all parameters for $0.6 < \alpha \leq 2$

\[
v_\alpha = \frac{x_{0.95} - x_{0.05}}{x_{0.75} - x_{0.25}} \quad v_\beta = \frac{x_{0.95} + x_{0.05} - 2x_{0.50}}{x_{0.95} - x_{0.05}}
\]

$$\alpha = \psi_1(v_\alpha, v_\beta)$$

- Table:

<table>
<thead>
<tr>
<th>$v_\alpha$</th>
<th>0.0</th>
<th>0.1</th>
<th>0.2</th>
<th>0.3</th>
<th>0.5</th>
<th>0.7</th>
<th>1.0</th>
</tr>
</thead>
<tbody>
<tr>
<td>2.439</td>
<td>2.000</td>
<td>2.000</td>
<td>2.000</td>
<td>2.000</td>
<td>2.000</td>
<td>2.000</td>
<td>2.000</td>
</tr>
<tr>
<td>2.5</td>
<td>1.916</td>
<td>1.924</td>
<td>1.924</td>
<td>1.924</td>
<td>1.924</td>
<td>1.924</td>
<td>1.924</td>
</tr>
<tr>
<td>2.6</td>
<td>1.808</td>
<td>1.813</td>
<td>1.829</td>
<td>1.829</td>
<td>1.829</td>
<td>1.829</td>
<td>1.829</td>
</tr>
<tr>
<td>2.7</td>
<td>1.729</td>
<td>1.730</td>
<td>1.737</td>
<td>1.745</td>
<td>1.745</td>
<td>1.745</td>
<td>1.745</td>
</tr>
<tr>
<td>2.8</td>
<td>1.664</td>
<td>1.663</td>
<td>1.663</td>
<td>1.668</td>
<td>1.676</td>
<td>1.676</td>
<td>1.676</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>$v_\beta$</th>
<th>0.0</th>
<th>0.1</th>
<th>0.2</th>
<th>0.3</th>
<th>0.5</th>
<th>0.7</th>
<th>1.0</th>
</tr>
</thead>
<tbody>
<tr>
<td>3.0</td>
<td>1.563</td>
<td>1.560</td>
<td>1.553</td>
<td>1.548</td>
<td>1.547</td>
<td>1.547</td>
<td>1.547</td>
</tr>
<tr>
<td>3.2</td>
<td>1.484</td>
<td>1.480</td>
<td>1.471</td>
<td>1.460</td>
<td>1.448</td>
<td>1.438</td>
<td>1.438</td>
</tr>
<tr>
<td>3.5</td>
<td>1.391</td>
<td>1.386</td>
<td>1.378</td>
<td>1.364</td>
<td>1.337</td>
<td>1.318</td>
<td>1.318</td>
</tr>
<tr>
<td>4.0</td>
<td>1.273</td>
<td>1.266</td>
<td>1.250</td>
<td>1.210</td>
<td>1.184</td>
<td>1.150</td>
<td>1.150</td>
</tr>
<tr>
<td>8.0</td>
<td>1.029</td>
<td>1.021</td>
<td>1.014</td>
<td>1.004</td>
<td>0.974</td>
<td>0.935</td>
<td>0.874</td>
</tr>
<tr>
<td>10.0</td>
<td>0.896</td>
<td>0.892</td>
<td>0.887</td>
<td>0.883</td>
<td>0.855</td>
<td>0.823</td>
<td>0.769</td>
</tr>
<tr>
<td>15.0</td>
<td>0.698</td>
<td>0.695</td>
<td>0.692</td>
<td>0.689</td>
<td>0.676</td>
<td>0.656</td>
<td>0.595</td>
</tr>
<tr>
<td>25.0</td>
<td>0.593</td>
<td>0.590</td>
<td>0.588</td>
<td>0.586</td>
<td>0.579</td>
<td>0.563</td>
<td>0.513</td>
</tr>
</tbody>
</table>
Characteristic function based methods

- For a sample $x_1, \ldots, x_n$ define sample cf:
  \[ \hat{\phi}(t) = \frac{1}{n} \sum_{j=1}^{n} e^{itx_j} \]

- Press (1972) – method of moments
  - for a choice of $t_1, \ldots, t_4$
    \[
    \hat{\alpha} = \frac{\log \frac{\log |\hat{\phi}(t_1)|}{\log |\hat{\phi}(t_2)|}}{\log \frac{t_1}{t_2}} \quad \hat{\beta} = \frac{\hat{u}(t_4) - \hat{u}(t_3)}{[t_4|\hat{\alpha} - 1| - t_3|\hat{\alpha} - 1|] \hat{\alpha} \tan \frac{\hat{\alpha} \pi}{2},}
    \]
    \[
    \log \hat{\sigma} = \frac{\log |t_1| \log(-\log |\hat{\phi}(t_2)|) - \log |t_2| \log(-\log |\hat{\phi}(t_1)|)}{\log |t_1|} \quad \hat{\mu} = \frac{|t_4|\hat{\alpha} - 1| \hat{u}(t_3) - t_3|\hat{\alpha} - 1| \hat{u}(t_4)|}{|t_4|\hat{\alpha} - 1| - t_3|\hat{\alpha} - 1|}.
    \]

- Press (1972), Leitch-Paulson (1975) – minimum distance method
  \[
  h(\alpha, \sigma, \beta, \mu) = \int_{-\infty}^{\infty} |\phi(t) - \hat{\phi}(t)|^r W(t) \, dt.
  \]
Characteristic function based methods: regression

  - From the definition of the cf in representation \( S_\alpha(\sigma,\beta,\mu) \):
    \[
    \log(-\log|\phi(t)|^2) = \log(2\sigma^\alpha) + \alpha \log |t|.
    \]
    \[
    \arctan\left(\frac{Im\phi(t)}{Re\phi(t)}\right) = \mu t + \beta \sigma^\alpha \tan \frac{\pi \alpha}{2} \operatorname{sign}(t)|t|^\alpha
    \]
  - Thus to estimate \( \alpha \) and \( \sigma \) we can:
    \[\text{regress} \quad y = \log(-\log|\phi_n(t)|^2) \text{ on } w = \log |t|\]
    \[y_k = m + \alpha w_k + \epsilon_k, \quad k = 1, 2, ..., K,\]

- **Kogon-Williams (1998)** used the \( S_\alpha^0(\sigma,\beta,\mu_0) \) representation and performed an initial scale-location normalization
Maximum Likelihood Method


\[
f_\alpha(x) = \frac{\alpha}{\pi |1 - \alpha|} x^{1/(\alpha - 1)} \int_0^{\pi/2} U_\alpha(\gamma, 0) e^{-x^{\alpha/(\alpha - 1)} U_\alpha(\gamma, 0)} d\gamma
\]

- STABLE (www.academic2.american.edu/~jpnolan/stable/stable.html) uses adaptive quadrature DQDAG to evaluate the integrals in the general asymmetric case

- XploRe 4.6 (www.xplore-stat.de) includes estimation methods
  - Net based data analysis
Choosing the best method

- **Fastest**
  - Quantile method
    - (McCulloch)
  - CF regression
    - (Koutrouvelis-Kogon-Williams)
- **Slowest**
  - MLE
    - (Nolan)

- **Least accurate**
- **Most accurate**
Topics

- Introduction
- Properties of stable laws
- Computer simulation of stable variables
- Estimation of parameters
- Other interesting topics
  - Multivariate $\alpha$-stable laws
  - $\alpha$-stable stochastic processes
Multivariate stable laws

- Bivariate stable fit to DEM/GBP and JPY/GBP (Nolan, 1999)

- McCulloch (1999)
- Davydov (2001)
$\alpha$-stable stochastic processes

- Ito-McKean (1965), Lukacs (1967), Breiman (1968) – $\alpha$-stable motion (i.e. process with $\alpha$-stable, independent and stationary increments – called an $\alpha$-stable Levy process in math literature)
- Rootzen (1978), Kokoszka-Taqqu (1994) – $\alpha$-stable (F)AR(I)MA
- Cambanis-Soltani (1984) – prediction of harmonizable processes
- Rosinski (1995) – structure of stationary $S\alpha S$ processes
Stable distributions and processes provide models for heavy-tailed data.
See:

- … and references therein.

These and related papers can be downloaded from:

- www.im.pwr.wroc.pl/~hugo/Publications.html