Stochastic volatility model of Heston and the smile

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Agenda

1. FX markets and the smile
2. Heston’s model
3. Calibration
FX markets

EUR/USD and USD/JPY are two of the most liquid underlying markets with trading in:

- Spot/forward (ca. 90% of activity, very small margins)
- Vanilla options (9%, small margins)
- Exotic options (1%, potentially high margins)
Global markets

USD/JPY market activity
Black-Scholes type formula

- Assumes that asset prices follow GBM:

$$dS_t = S_t(\mu dt + \sigma dB_t)$$  \hspace{1cm} (1)

- European FX call option price

(Garman and Kohlhagen, 1983):

$$C_t = S_t e^{-rf\tau} \Phi(d_1) - Ke^{-r\tau} \Phi(d_2),$$

where

$$d_1 = \frac{\log(S_t/K) + (r-r_f + \frac{1}{2} \sigma^2)\tau}{\sigma \sqrt{\tau}}, \quad d_2 = d_1 - \sigma \sqrt{\tau}$$
BS formula is flawed

- Implied volatility $\sigma_i$ is the volatility that equates the BS price:
  \[ \text{BS}(S_t, K, r, \sigma_i, \tau) = \text{Option market price} \]
- Model implied volatilities for different strikes and maturities are not constant
- Volatility smile or smirk/grin is observed
The smile

1W, 1M, 3M, 6M, 1Y, 2Y EUR/USD smiles

1W (black), 1M (red), 3M (green), 6M (blue), 1Y (cyan), and 2Y (yellow) EUR/USD implied volatility smiles on July 1, 2004
The smirk/grin

ODAX options implied $\sigma$’s on Oct. 12, 1998

- $\tau = 10$ days
- $\tau = 38$ days
- $\tau = 66$ days
- $\tau = 157$ days
Correcting the BS formula (1/3)

- Allow the volatility to be a deterministic function of time (Merton, 1973): \( \sigma = \sigma(t) \)
- Explains the different \( \sigma_i \) levels for different \( \tau \)'s, but cannot explain the smile shape for different strikes
Correcting the BS formula (2/3)

- Allow not only time, but also state dependence of $\sigma$ (Dupire, 1994; Derman and Kani, 1994; Rubinstein, 1994): $\sigma = \sigma(t, S_t)$

- Lets the local volatility surface to be fitted, but cannot explain the persistent smile shape which does not vanish as time passes
Correcting the BS formula (3/3)

- Allow the volatility coefficient in the BS diffusion equation (1) to be random: $\sigma = \sigma_t$

- Pioneering work of Hull and White (1987), Stein and Stein (1991), and Heston (1993) led to the development of stochastic volatility models
Heston’s model

\[ dS_t = S_t \left( \mu \, dt + \sqrt{v_t} \, dW^{(1)}_t \right), \quad (2) \]

\[ dv_t = \kappa (\theta - v_t) \, dt + \sigma \sqrt{v_t} \, dW^{(2)}_t, \quad (3) \]

\[ dW^{(1)}_t \, dW^{(2)}_t = \rho \, dt \quad (4) \]

- Variance process (3) is non-negative and mean-reverting (as observed in the markets)
- It has CIR dynamics
**GBM vs. Heston**: $\rho = -0.05$, initial (=GBM) variance $v_0 = 4\%$, long term var. $\theta = 4\%$, speed of mean reversion $\kappa = 2$, vol of vol $\sigma = 30\%$
Unlike Gaussian tails, tails of Heston’s marginals are exponential: log-densities resemble hyperbolas (Dragulescu and Yakovenko, 2002)
Option pricing in Heston’s model

- PDE for the option price can be solved analytically using the method of characteristic functions (Heston, 1993)

- Closed-form solution for vanilla options:

\[
 h(t) = HestonVanilla(\kappa, \theta, \sigma, \rho, \lambda, r_d, r_f, v_0, S_0, K, \tau) \\
= e^{-r_f \tau} S_t P_+(\phi) - K e^{-r_d \tau} P_-(\phi) 
\] (5)
Calibration

1. Look at a time series of historical data:
   - Use GMM, SMM, EMM, or Bayesian MCMC to fit the price process
   - Fit empirical distributions of returns to the marginal distributions
   - Cannot estimate the market price of risk $\lambda$

2. Calibrate the model to derivative prices or better to the volatility smile
Calibration to the smile

- Take the smile of the current vanilla options market as a given starting point
- Find the optimal set of model parameters for a fixed $\tau$ and a given vector of market BS implied volatilities $\{\hat{\sigma}_i\}_{i=1}^n$ for a given set of delta pillars $\{\Delta_i\}_{i=1}^n$
- No need to worry about estimating $\lambda$ as it is already embedded in the market smile
EUR/USD volatility surface on July 1, 2004: the fit is very good for maturities between three and eighteen months
Unfortunately, Heston’s model does not perform satisfactorily for short maturities and extremely long maturities.
Only 3 parameters to fit

\[ \theta(\approx v_0) - \text{ATM level of the smile} \]

\[ \rho - \text{skew (quoted as risk reversals)} \]

\[ \sigma(\approx \kappa) - \text{convexity (butterflies)} \]
Application

1. Calibrate the model to vanilla options

2. Employ it for pricing exotics, like one-touch or barrier options (finite difference, Monte Carlo)
References


http://www.xplore-stat.de/ebooks/ebooks.html